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TECHNICAL REPORT

The main, long term, objective of this work is research and development of the theory and constructive computational algorithms for synthesis of single and dual reflector antenna systems. The work is based on analytic and numerical procedures for solving the underlying nonlinear boundary value problems.

To this end we have been carrying out investigations in the following directions:

1. Conditions for solvability of the direct problem of reconstructing the reflector antenna with uniform output density of the reflected rays from the given far field, input aperture, and prescribed in advance (non-radially symmetric) input density.

2. Properties of reciprocal reflector and connections between the direct and inverse problems.

3. Construction and testing of an algorithm based on a "diffusion" - type scheme for solving numerically the boundary value problem formulated in 1.

4. Formulation of the dual reflector problem as direct and inverse boundary value problems and investigation of appropriate solvability conditions.

In item 1 we succeeded in deriving the equation in concise and explicit form. As far as we know, in such form this equation is obtained for the first time. All the terms in the equation have a simple and clear geometric meaning and can be computed numerically by efficient numerical procedures. Our derivation is based on a general procedure utilizing differential geometric methods. We also use the same methods for deriving the equations of the dual reflector problem in item 4. We have also shown, in explicit form, the connection between the inverse and direct problems in the single reflector setting (item 2). Using this connection we established conditions for solvability of direct problem. Regarding item 3 we developed and tested an algorithm for solving numerically the direct single reflector problem. Our approach here is via a certain special "diffusion" -type procedure. With this approach it is possible to avoid a costly numerical inversion of a highly nonlinear second order differential operator.

The results are being organized in a series of papers. Two of them "Differential-geometric methods in design of single and dual reflector antennas" and "On one direct problem in the reflector antenna theory" are complete and submitted for publication. Copies of both papers are attached to this report. The third paper "On the theory of dual offset reflector antennas" is being prepared for publication.

Differential-geometric methods in design of single and dual reflector antennas[1]

by

Vladimir Oliker, Elsa Newman[2], and Laird Prussner

Introduction

This paper is the first in a series of three papers in which we study the theory and numerical methods in synthesis of reflector antennas.

The problem of synthesizing a single or dual reflector antenna system producing a pre-specified intensity distribution on a far-field or on a target domain continues to attract considerable attention of researchers and practitioners. In the geometric optics approximation (GO) the basic laws of reflection can be used to derive a system of three first order partial differential equations (PDE's) corresponding to the problem. This was done in the 60s by B. Kinber [7] and V. Galindo-Israel [4]. This system of PDE's, roughly speaking, consists of two "parts": two equations in the system express the Snell law, and the third equation relates the output intensity distribution to the intensity of the primary source. The latter is a strongly nonlinear equation which is essentially a condition on the Jacobian of the "reflector" map transforming the input spherical wave front into the output front.

It is well known (see, for example, [3], Ch. I) that a first order system of PDE's admits a smooth solution only if certain integrability conditions are satisfied. It was observed by Kinber [7] and Galindo-Israel and Mittra (see [5] and further references there) that for the first order PDE's system describing reflector antennas these integrability conditions, in general, may not be fulfilled. In fact, it is not difficult to write down explicitly the integrability conditions for the two equations

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[1] This research was supported by AFOSR under contract F499620-91-C-0001. The United States Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation herein.

[2] The author is a graduate student.

The order of the last two authors is arbitrary

expressing Snell's law; see [5], [6]. However, this is not sufficient, since the third equation, relating the input and output intensities, is not taken into account. Consequently, one can not expect, in general, that the problem will have a solution.

The exceptional situation when the integrability conditions for the entire system are always satisfied is the case of axially symmetric reflectors and axially symmetric data. It has been known for some time and was shown rigorously in [15] that in this case one can give explicit necessary and sufficient conditions for solvability of the problem. In the subsequent paper we will show that if the input domain and the far-field are coaxial and there is an axially symmetric solution of the problem with axially symmetric intensities then one can always find a solution of the first order PDE's system for prescribed non-axially symmetric intensities close in some norm to the axially symmetric one.

In reflector antenna systems with two or more reflecting surfaces there are additional degrees of freedom to control the output amplitude and phase. However, the more reflecting surfaces are in a system the larger the number of PDE's describing it. Consequently, the complexity of the system increases and its analysis becomes harder. As far as we know up to present there are no rigorous results concerning such systems. The authors usually use numerical computations to obtain some acceptable variant of a "solution" (even when a true solution may not exist). In the most recent works of K. Shogen, R. Mittra, V. Galindo-Israel and W.A. Imbriale various difficulties in numerical treatment of these equations are reported [17].

In 1972 J. Schruben [16] considered the problem of designing a lighting fixture which would produce a pre-specified intensity pattern on a plane aperture. In her approach, the reflector surface is described as a graph of a scalar function over a domain on a unit sphere (the input aperture) and the Jacobian of the reflector map is expressed in terms of this function and its derivatives up to second order. The corresponding expression is a second order nonlinear PDE of Monge-Ampere type. Unfortunately, the use of a specific coordinate system made the formulas in [16] quite complicated, and Schruben does not even derive explicitly the equation of the problem. Her main concern was to describe conditions for ellipticity of the operator involved.
The principal advantage in this approach is that instead of a system one

has to deal here with one equation for one scalar function. This equation, however, is of second order and strongly nonlinear.

In 1974 B. Westcott et al. initiated a different approach to synthesis of reflectors. In case of a single reflector, required to produce a prescribed far-field pattern, this approach to the problem corresponds to formulating it as an inverse problem. It is based on the observation that the reflector can be parametrized in a special way by points on the far-field (or target) and actually recovered from one scalar function. The latter must also satisfy a second order PDE of Monge-Ampere type relating the intensity of the primary source and of the desired output. This approach was pursued by Westcott and his colleagues in a number of publications; see [11], [1], [2], [19], [18], and further references there.

In all of these approaches, rigorous general results concerning existence and uniqueness of solutions are lacking. Eventually, the authors always have to resort to numerical calculations. Some progress towards establishing rigorous uniqueness results was made by Marder [10], who considered the single reflector problem in the setting of the approach by Westcott et al. Also, in the inverse problem setting rigorous results establishing existence of non-axially symmetric solutions with prescribed non-axially symmetric densities, close (in a certain norm) to axially symmetric, were obatained by Oliker [13] (see also [14] for related results) in the case when the problem is treated.

In this paper, we study the "direct" problem of synthesizing the reflector surface. Our starting point is similar to that of Schruben in the sense that we also describe the sought reflector surface as a graph over a spherical domain (the input aperture) and look for a second order PDE which the function describing the graph satisfies. However, in contrast with the paper of Schruben, we succeed in deriving such an equation in concise and explicit form. This is done in both cases of single and dual reflector antennas. In this paper we present our results for a single reflector antenna. The results for dual reflector antennas will be presented in a subsequent publication.

The corresponding expressions are relatively simple and, most importantly, contain familiar geometric quantities for which various stable procedures for numerical computation are available. Our derivations are based on differential geometric methods. We also show that in the case of a single reflector surface our equation and the one

derived by Brickell-Marder-Westcott [2] are connected by a simple transformation. In fact, we show that the reflector surfaces described in the direct problem and the reflector surfaces constructed in the inverse problem are actually what is known as reciprocal reflectors. In the complex analytic formulation, such a connection between the direct and inverse problems was established in [2]. Our approach to establishing this connection does not involve complex analysis. In the real-valued form this connection becomes quite transparent and follows easily from the formulas we develop. We also discuss the question of optimal boundary conditions to be imposed on the solution. This question is important. As we pointed out earlier, the problem, in general, may be overdetermined and therefore lack a solution.

The paper is organized as follows. In section 1 we review some facts from elementary differential geometry and develop basic geometric formulas for reflecting surfaces. In section 2 we compute the Jacobian of the "reflector" map and relate it to the input and output power densities. The main result here is the formula (2.12) expressing the Jacobian in geometric quantities. As far as we know, in such explicit form this expression appears for the first time. This expression turned out to be very useful. In section 3 we use it to construct the reciprocal reflector and show the connection between the direct and inverse formulations of the problem. In section 4 we present and analyze in our framework the formulations of the problem as a first order system of PDE's and as a boundary value problem for a second order nonlinear PDE of Monge-Ampere type. In section 5 we give a rigorous treatment of the axially symmetric case of the direct problem. In section 6 a numerical algorithm based on formula (2.12) is presented.

1. Preliminaries

1.1. In three dimensional space $\mathbf{R}^3$ let S be a unit sphere centered at some point O. Fix a Cartesian coordinate system with the origin O. Let Ω be a domain on S, $\overline{\Omega}$ its closure, and $\mathbf{m}$ a unit vector with endpoint in $\overline{\Omega}$. Let ρ be a positive function in $\overline{\Omega}$ and set $\mathbf{r}(\mathbf{m}) = \rho(\mathbf{m})\mathbf{m}$. Then $\mathbf{r}$ defines a surface F projecting radially from O univalently onto $\overline{\Omega}$. Denote by $\mathbf{n}$ the unit normal vector field on F and assume that F is

oriented so that $<\mathbf{m},\mathbf{n}> > 0$ everywhere on F. An illustration of this situation is given on the figure on p. 8.

We now recall some basic notions from Differential Geometry that will be needed later. These can be found, for example, in E. Kreyszig [9].

Let (u^1,u^2) be some smooth local coordinates on S. Put $u = (u^1,u^2)$. Then $\mathbf{m} = \mathbf{m}(u)$ $(\equiv \mathbf{m}(u^1,u^1))$ is a vector valued function of u giving the position vector of any point in $\overline{\Omega} \subset S$. For that reason $\mathbf{m}(u)$ is viewed as a unit vector in $\mathbf{R}^3$ and also as a point in $\overline{\Omega}$. As usual, we put $f(\mathbf{m}(u)) \equiv f(u^1,u^1)$ for any function $f:S \to \mathbf{R}$. Everywhere in the paper, we use the following range of indices: $1 \le i,j,k,... \le 2$.

The first fundamental form $e = e_{ij}du^i du^j$ of S has coefficients

$$e_{ij} = < \mathbf{m}_i,\mathbf{m}_j>$$

where $< , >$ denotes the scalar product in $\mathbf{R}^3$ and $\mathbf{m}_i = \partial\mathbf{m}/\partial u^i$. Here and everywhere below the summation convention over repeated lower and upper indices is in effect. The matrix $[e_{ij}]$ is symmetric and invertible; its inverse is denoted by $[e^{ij}]$.

We assume that the coordinates u^1, u^2 are chosen so that $<\mathbf{m},\mathbf{m}_1 \times \mathbf{m}_2> > 0$ in $\overline{\Omega}$.

The area element $d\sigma$ of S (in $\overline{\Omega}$) is given by

$$d\sigma = \left|\mathbf{m}_1 \times \mathbf{m}_2\right| du^1du^2 = \langle \mathbf{m},\mathbf{m}_1 \times \mathbf{m}_2 \rangle du^1du^2 = \sqrt{\det(e_{ij})}$$

The first fundamental form $g = g_{ij}du^i du^j$ of the reflector surface F has coefficients

$$g_{ij} = <\mathbf{r}_i,\mathbf{r}_j> = \rho_i\rho_j + \rho^2 e_{ij}$$

where $\mathbf{r}_i = \partial\mathbf{r}/\partial u^i$ and $\rho_i = \partial\rho/\partial u^i$. Additionally, we set

$$\nabla\rho = \rho_i g^{ij}\mathbf{r}_j, \quad \tilde{\nabla}\rho = \rho_i e^{ij}\mathbf{m}_j$$

where $[g^{ij}] = [g_{ij}]^{-1}$. Then

$$|\nabla\rho|^2 \equiv <\nabla\rho,\nabla\rho> = \rho_i\rho_j g^{ij},\ |\tilde{\nabla}\rho|^2 \equiv <\tilde{\nabla}\rho,\tilde{\nabla}\rho> = \rho_i\rho_j e^{ij}.$$

The vectors $\mathbf{m}_1(u)$, $\mathbf{m}_2(u)$, and $\mathbf{m}(u)$ form a basis of $\mathbf{R}^3$ at every $u \in \overline{\Omega}$ and any vector in $\mathbf{R}^3$ can be expressed in terms of this "moving" basis. In particular, we may express the unit normal vector field $\mathbf{n}(u)$ on the reflector surface F as

$$\mathbf{n} = \frac{\rho\mathbf{m} - \tilde{\nabla}\rho}{\sqrt{\rho^2 + |\tilde{\nabla}\rho|^2}} \tag{1.1}$$

Since $\mathbf{r}_i = \rho_i\mathbf{m} + \rho\mathbf{m}_i$, we have $<\mathbf{r}_i,\mathbf{n}> = 0$. Obviously, $|\mathbf{n}| = 1$ and so $\mathbf{n}$ is indeed the unit normal field on F.

It follows from (1.1) that

$$\langle\mathbf{r},\mathbf{n}\rangle = \rho\langle\mathbf{m},\mathbf{n}\rangle = \frac{\rho^2}{\sqrt{|\tilde{\nabla}\rho|^2 + \rho^2}}. \tag{1.2}$$

Since by our assumption $<\mathbf{m},\mathbf{n}> > 0$ and $|\mathbf{r}| = \rho > 0$ on F, we see that

$$<\mathbf{r},\mathbf{n}> > 0 \text{ on F}. \tag{1.3}$$

The covariant derivatives relative to the first fundamental form of F are defined for any scalar function f: F->$\mathbf{R}$ as follows:

$$\nabla_i f \equiv f_i,$$

$$\nabla_{ij} f \equiv f_{ij} - \Gamma^k_{ij} f_k,$$

where

$$f_i = \frac{\partial f}{\partial u^i},\ f_{ij} = \frac{\partial^2 f}{\partial u^i \partial u^j},$$

and the Γ's are the Christoffel symbols associated with g.

Similarly, for $\overline{\Omega} \subset S$ and $f\colon \overline{\Omega} \to \mathbf{R}$, the covariant differentiation is defined as

$$\tilde{\nabla}_i f \equiv f_i,$$
$$\tilde{\nabla}_{ij} f \equiv f_{ij} - \tilde{\Gamma}^k_{ij} f_k,$$

where the $\tilde{\Gamma}$'s represents the Christoffel symbols associated with the first fundamental form e of S.

If $\mathbf{v} = (v_1,v_2,v_3)$ is a vector field on F then

$$\nabla_i \mathbf{v} = (\nabla_i v_1, \nabla_i v_2, \nabla_i v_3)$$

and similarly one defines $\nabla_{ij}\mathbf{v}$, and $\tilde{\nabla}_i \mathbf{v}$, $\tilde{\nabla}_{ij}\mathbf{v}$ if $\mathbf{v}$ is a vector field on $\overline{\Omega} \subset S$. The coefficients of the second fundamental form $b = b_{ij}du^i du^j$ on the surface F are given by

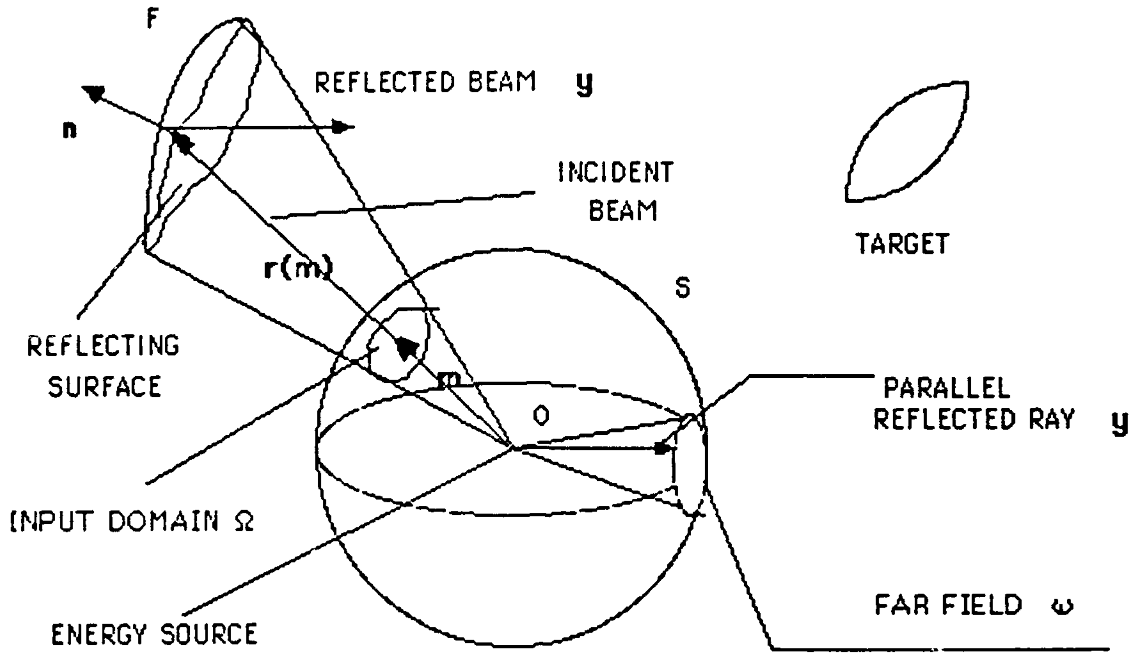
$$b_{ij} = \langle \mathbf{r}_{ij}, \mathbf{n} \rangle = -\langle \mathbf{r}_i, \mathbf{n}_j \rangle$$

where $\mathbf{r}_{ij} = \partial^2 \mathbf{r}/\partial u^i \partial u^j$.

According to the classical derivation formulas

$$\mathbf{r}_{ij} = \Gamma^k_{ij}\mathbf{r}_k + b_{ij}\mathbf{n} \qquad (1.4)$$

1.2 Suppose a light ray is originated at O in the direction $\mathbf{m}$ and is reflected at the point $\mathbf{r}(\mathbf{m})$ in the direction $\mathbf{y}$.



By Snell's law,

$$\mathbf{y} = \mathbf{m} - 2 \langle \mathbf{m},\mathbf{n} \rangle \mathbf{n},$$

and we may consider the map $\gamma: \overline{\Omega} \to S$, $\gamma(\mathbf{m}) = \mathbf{y}(\mathbf{m})$, $\mathbf{m} \in \overline{\Omega}$. Put $\overline{\omega} = \gamma(\overline{\Omega})$. The may γ transforms $\overline{\Omega} \to \overline{\omega}$. Consequently, we can relate the area elements in $\overline{\Omega}$ and in its image $\overline{\omega}$ by computing the Jacobian of the map γ. Note that since $\mathbf{y}(\mathbf{m})$ is the unit vector in the reflected direction, the Jacobian $J(\mathbf{m}(u))$ is given by

$$J(\mathbf{m}(u)) = \pm \frac{|d\gamma(\mathbf{m}(u))|}{|d\sigma(u)|} = \pm \frac{\sqrt{\det\left\langle \mathbf{y}_i(u),\mathbf{y}_j(u) \right\rangle}}{\sqrt{\det\left\langle \mathbf{m}_i(u),\mathbf{m}_j(u) \right\rangle}} \tag{1.5}$$

We assign a ± sign to the Jacobian according to whether γ preserves the orientation or reverses it. This is equivalent to considering the volumes' relationship

$$J(\mathbf{m}) \langle \mathbf{m},(\mathbf{m}_1 \times \mathbf{m}_2) \rangle = \langle \mathbf{y},(\mathbf{y}_1 \times \mathbf{y}_2) \rangle .$$

Since $\langle \mathbf{m}, \mathbf{m}_1 \times \mathbf{m}_2 \rangle > 0$, J is positive if $\langle \mathbf{y}, \mathbf{y}_1 \times \mathbf{y}_2 \rangle > 0$ and J is negative otherwise. Note that $|\langle \mathbf{y},(\mathbf{y}_1 \times \mathbf{y}_2)\rangle| = |\det\langle \mathbf{y}_i,\mathbf{y}_j\rangle|$.

1.3 We will need several different expressions for the vector functions $\mathbf{m}(u)$ and $\mathbf{y}(u)$. First, we find an expression for $\mathbf{m}(u)$ in terms of the basis $\mathbf{r}_1(u)$, $\mathbf{r}_2(u)$, $\mathbf{n}(u)$. These vectors indeed form a basis since

$$|\mathbf{r}_1(u) \times \mathbf{r}_2(u)| = \rho^2(u)\{|\tilde{\nabla}\rho(u)|^2 + \rho^2(u)\}\det(e_{ij}(u)) > 0,$$

and $\mathbf{n}(u)$ is perpendicular to $\mathbf{r}_1(u)$ and $\mathbf{r}_2(u)$.

Now, expressing $\mathbf{m}(u)$ in terms of $\mathbf{r}_1(u)$, $\mathbf{r}_2(u)$, and $\mathbf{n}(u)$, we find (omitting the argument u)

$$\mathbf{m} = \rho_i g^{ij}\mathbf{r}_j + \sqrt{1 - |\nabla\rho|^2}\,\mathbf{n} \tag{1.6}$$

Obviously, this formula is valid only if $|\nabla\rho|^2 \leq 1$. However, it follows from (1.2) and (1.6) that

$$\langle \mathbf{m},\mathbf{n}\rangle = \frac{\rho}{\sqrt{|\tilde{\nabla}\rho|^2 + \rho^2}} = \sqrt{1 - |\nabla\rho|^2}. \tag{1.7}$$

Since it is always assumed that $\langle \mathbf{m},\mathbf{n}\rangle > 0$ and $\rho > 0$ on F, the condition $|\nabla\rho|^2 < 1$ is fulfilled.

We may express $\mathbf{y}$ in terms of ρ, $\mathbf{m}$, and their derivatives. It follows from Snell's law and (1.1) that

$$\mathbf{y} = \mathbf{m} - 2\,\frac{\rho(\rho\mathbf{m} - \tilde{\nabla}\rho)}{|\tilde{\nabla}\rho|^2 + \rho^2} \tag{1.8}$$

Using Snell's law and (1.7), we may express $\mathbf{y}$ without explicit use of $\mathbf{m}$:

$$\mathbf{y} = \rho_i g^{ij}\mathbf{r}_j - \sqrt{1 - |\nabla\rho|^2}\,\mathbf{n} \tag{1.9}$$

2. Computation of the Jacobian of the map γ and the "balance" equation

2.1. Here we find an explicit expression for the Jacobian J(**m**). As it follows from (1.5), we need an expression for $\det\langle \mathbf{y}_i, \mathbf{y}_j \rangle$.

Proposition 2.1. Let F be a reflector surface as in section 1. Put

$$H_{ij} = \nabla_{ij}\rho + \sqrt{1 - |\nabla\rho|^2}\, b_{ij} \tag{2.1}$$

$$\rho^s = g^{sk}\rho_K .$$

Then

$$\sqrt{\det\langle y_i, y_j \rangle} = \frac{|\det H_{ij}|}{\sqrt{\det g_{ij}}} \frac{1}{\sqrt{1 - |\nabla\rho|^2}} . \tag{2.2}$$

Remark. Note that since $|\nabla\rho|^2 < 1$, (2.2) is well defined.

Proof. We begin by showing that

$$\langle \mathbf{y}_i, \mathbf{y}_j \rangle = H_{ik}H_{js}\left(g^{ks} + \frac{\rho^k\rho^s}{1 - |\nabla\rho|^2} \right). \tag{2.3}$$

Using (1.9), we differentiate $\mathbf{y}$ covariantly relative to the form g and obtain

$$\nabla_i\mathbf{y} = \mathbf{y}_i = \nabla_{ki}\rho g^{kj}\mathbf{r}_j + \frac{g^{kj}\nabla_{ki}\rho\rho_j}{\sqrt{1 - |\nabla\rho|^2}}\mathbf{n} - \sqrt{1 - |\nabla\rho|^2}\,\mathbf{n}_i .$$

To simplify this expression we use (1.4) and the Weingarten equations [9], p. 126,

$$\mathbf{n}_i = -\, b_{ij} g^{jk} \mathbf{r}_k \,.$$

Now we see that

$$\begin{aligned}\mathbf{y}_i &= \left[\nabla_{ji}\rho + \sqrt{1-|\nabla\rho|^2}\, b_{ji}\right] g^{jk}\mathbf{r}_k \\ &\qquad + \left[\nabla_{ji}\rho + \sqrt{1-|\nabla\rho|^2}\, b_{ji}\right] g^{jk}\, \frac{\rho_k \mathbf{n}}{\sqrt{1-|\nabla\rho|^2}} \\ &= \left[\nabla_{ji}\rho + \sqrt{1-|\nabla\rho|^2}\, b_{ji}\right] g^{jk}\left[\mathbf{r}_k + \frac{\rho_k}{\sqrt{1-|\nabla\rho|^2}}\mathbf{n}\right],\end{aligned}$$

and hence

$$\mathbf{y}_i = H_{ji} g^{jk}\left(\mathbf{r}_k + \frac{\rho_k}{\sqrt{1-|\nabla\rho|^2}}\mathbf{n}\right). \tag{2.4}$$

Calculating $\langle \mathbf{y}_i, \mathbf{y}_j \rangle$, we immediately arrive at (2.3).

On the other hand, we have

$$\det\left(g^{ij} + \frac{\rho^i \rho^j}{1-|\nabla\rho|^2}\right) = \frac{1}{\det(g_{ij})\left(1-|\nabla\rho|^2\right)}. \tag{2.5}$$

Consequently, evaluating the determinants on both sides of (2.3), we obtain (2.2). The proposition is proved.

2.2. It will be useful to find an expression of (2.2) in terms of ρ and its derivatives on S.

Proposition 2.2. Let H_{ij}, b_{ij}, and e_{ij} be as before. Then

$$H_{ij} = 2b_{ij}\langle \mathbf{m},\mathbf{n}\rangle + \rho e_{ij} \tag{2.6}$$

Proof. It follows directly from (1.6) that

$$H_{ij} = 2b_{ij}\langle m,n\rangle + \nabla_{ij}\rho \tag{2.7}$$

Since $\rho^2 = \langle \mathbf{r},\mathbf{r}\rangle$, and $\rho\rho_i = \langle \mathbf{r},\mathbf{r}_i\rangle$, we differentiate covariantly and obtain

$$\rho_i\rho_j + \rho\nabla_{ij}\rho = \langle \mathbf{r}_i,\mathbf{r}_j\rangle + \langle \mathbf{r},\nabla_{ij}\mathbf{r}\rangle = g_{ij} + b_{ij}\langle \mathbf{r},\mathbf{n}\rangle.$$

Hence, $\rho\nabla_{ij}\rho = b_{ij}\langle \mathbf{r},\mathbf{n}\rangle - \rho_i\rho_j + g_{ij}$. But

$$g_{ij} = \langle \mathbf{r}_i,\mathbf{r}_j\rangle = \langle \rho_i\mathbf{m} + \rho\mathbf{m}_i,\rho_j\mathbf{m} + \rho\mathbf{m}_j\rangle = \rho_i\rho_j + \rho^2 e_{ij}.$$

Thus,

$$\rho\nabla_{ij}\rho = b_{ij}\langle \mathbf{r},\mathbf{n}\rangle + \rho^2 e_{ij}.$$

Dividing by ρ and substituting in (2.7), we obtain (2.6). The proposition is proved.

The following expression for the second fundamental form in terms of ρ is derived in [12]

$$b_{ij} = \frac{\rho\tilde{\nabla}_{ij}\rho - \rho^2 e_{ij} - 2\rho_i\rho_j}{\sqrt{\rho^2 + |\tilde{\nabla}\rho|^2}} \tag{2.8}$$

Combining it with (2.6) and (1.2), we get the following expression

$$\rho H_{ij} = \frac{2\left[\rho\tilde{\nabla}_{ij}\rho - \rho^2 e_{ij} - 2\rho_i\rho_j\right]}{\sqrt{\rho^2 + |\tilde{\nabla}\rho|^2}}\,\frac{\rho^2}{\sqrt{\rho^2 + |\tilde{\nabla}\rho|^2}} + \rho^2 e_{ij}. \tag{2.9}$$

2.3. We now express $\det(g_{ij})$ in terms of $\det(e_{ij})$. Recalling that $g_{ij} = \langle r_i, r_j \rangle = \rho_i \rho_j + \rho^2 e_{ij}$ and evaluating the determinant of g_{ij}, we find

$$\det g_{ij} = \rho^2(\rho^2 + |\tilde{\nabla}\rho|^2)\det(e_{ij}). \tag{2.10}$$

We use equations (2.6), (2.10), and (1.7) to rewrite the equation (2.2) as

$$\sqrt{\det\langle \mathbf{y}_i, \mathbf{y}_j \rangle} = \frac{1}{\rho^2} \frac{|\det(2b_{ij}\langle \mathbf{m}, \mathbf{n} \rangle + \rho e_{ij})|}{\sqrt{\det(e_{ij})}}. \tag{2.11}$$

2.4. Summarizing the preceding results, we may now obtain the expression for the Jacobian of the map γ: $\overline{\Omega} \rightarrow \overline{\omega}$. Namely, taking into account the sign convention, we obtain from (1.5) and (2.11)

$$J = \pm \frac{\sqrt{\det\langle \mathbf{y}_i, \mathbf{y}_j \rangle}}{\sqrt{\det\langle \mathbf{m}_i, \mathbf{m}_j \rangle}} = \frac{1}{\rho^2} \frac{\det(2b_{ij}\langle \mathbf{m}, \mathbf{n} \rangle + \rho e_{ij})}{\det(e_{ij})} \equiv G(\rho). \tag{2.12}$$

We use expressions (2.8) and (2.14) to find

$$J = \frac{1}{(\rho^2 + |\tilde{\nabla}\rho|^2)} \frac{\det\{2\rho \tilde{\nabla}_{ij}\rho - (\rho^2 - |\tilde{\nabla}\rho|^2)e_{ij} - 4\rho_i \rho_j\}}{\det(e_{ij})} \tag{2.13}$$

2.5. Using the expression for $J(\mathbf{m})$, we are now in a position to relate the energy of the input primary source emitting a power density $I(\mathbf{m})$ to the desired output power pattern $V(\mathbf{y})$. Namely, if $|d\gamma(\mathbf{m})|$ is the area element in ω expressed via the map γ, then we have the point-wise balance equation

$$V(\mathbf{y}(\mathbf{m}))\ |d\gamma(\mathbf{m})| = I(\mathbf{m})d\sigma(\mathbf{m}),\ \mathbf{m} \in \Omega \tag{2.14}$$

or by (1.5)

$$V(\mathbf{y}(\mathbf{m}))\,|J(\mathbf{m})| = I(\mathbf{m}), \ \mathbf{m} \in \Omega. \tag{2.15}$$

Taking into account the sign of the Jacobian we obtain from (2.15) and (2.12)

$$V(\mathbf{y}(\mathbf{m}))\,\frac{\det\left(2\dfrac{b_{ij}(\mathbf{m})\langle \mathbf{m},\mathbf{n}(\mathbf{m})\rangle}{\rho(\mathbf{m})} + e_{ij}(\mathbf{m})\right)}{\det(e_{ij})} = I(\mathbf{m}), \quad \mathbf{m} \in \Omega\,. \tag{2.16}$$

Integrating (2.15) over Ω, we obtain

$$\int_\Omega V(\mathbf{y}(\mathbf{m}))\,|J(\mathbf{m})|\,d\sigma = \int_\Omega I(\mathbf{m})d\sigma.$$

Applying to the integral on the left the known formula for change of variables, we obtain

$$\int_{\omega=\gamma(\Omega)} V(\mathbf{y})\,|d\gamma| = \int_\Omega I(\mathbf{m})d\sigma \tag{2.17}$$

This formula expresses the energy conservation property of the reflector system. It is known in the literature as the "balance" equation; see [5], [1].

3. The reciprocal reflector

3.1. The representation of F as a graph of a function ρ over the "input" domain $\overline{\Omega}$ allows the construction of another reflecting surface F^* naturally associated with F. The surface F^* is constructed so that $\overline{\Omega}$ becomes the far-field domain while $\overline{\omega}$ becomes the "input" domain.

Define the surface F^* by the map

$$-\,\mathbf{r}^* = \tilde{\nabla}p + (p-\eta)\mathbf{m}, \ \mathbf{m} \in \overline{\Omega}, \tag{3.1}$$

$$p = 1/\rho,\ \eta = (p^2 + |\tilde{\nabla} p|^2)/(2p). \tag{3.2}$$

It is shown in [15] that $\mathbf{r}^*$ has the following properties:

(i) $\mathbf{r}_i^* := \partial\mathbf{r}^*/\partial u^i = q_{ij}e^{jk}(\mathbf{m}_k - (p_k/p)\mathbf{m}),\ i=1,2,$

where $q_{ij} = \tilde{\nabla}_{ij}p + (p-\eta)e_{ij}$;

(ii) the vector field

$$\mathbf{N} = -\frac{\tilde{\nabla} p + p\mathbf{m}}{\sqrt{p^2 + |\tilde{\nabla} p|^2}}, \quad \mathbf{m} \in \Omega,$$

satisfies the relation $\langle \mathbf{r}_i^*, \mathbf{N}\rangle = 0$, and therefore, if $\mathbf{r}^*$ is an immersion, then $\mathbf{N}$ is the unit normal vector field on F^*;

(iii) put $\boldsymbol{\xi}: \overline{\Omega} \to S^2$, $\boldsymbol{\xi}(\mathbf{m}) = \mathbf{r}^*(\mathbf{m})/\eta(\mathbf{m})$. Then obviously $\boldsymbol{\xi}$, $\mathbf{N}$, and $\mathbf{y}$ are coplanar and $-\langle\boldsymbol{\xi},\mathbf{N}\rangle = \langle\mathbf{m},\mathbf{N}\rangle$, that is, the law of reflection is satisfied. However, this time the reflector F^* is parametrized by points in $\overline{\Omega}$.

Explicit computations give:

$$\eta = \frac{\rho^2 + |\tilde{\nabla}\rho|^2}{2\rho^3}, \tag{3.3}$$

$$-\mathbf{r}^* = -\frac{\tilde{\nabla}\rho}{\rho^2} + \frac{\rho^2 - |\tilde{\nabla}\rho|^2}{2\rho^3}\mathbf{m}, \tag{3.4}$$

3.2. Proposition. Suppose the map $\boldsymbol{\xi}$ defined by F^* is a diffeomorphism of $\overline{\Omega}$ onto some $\overline{\Omega}' \subset S$. Assume that the map $\gamma: \overline{\Omega} \to \overline{\omega}$ is a diffeomorphism. Then, $\gamma(\mathbf{m}) = \boldsymbol{\xi}(\mathbf{m})$ for all $\mathbf{m} \in \overline{\Omega}$, and, consequently, $\overline{\Omega}' = \overline{\omega}$.

Proof. Since **m** is a reflected direction relative to F^*, we have by Snell's law

$$\mathbf{m} = \boldsymbol{\xi} - 2\langle\boldsymbol{\xi},\mathbf{N}\rangle\mathbf{N}. \tag{3.6}$$

We compute, using (3.3) and (3.5),

$$\eta\langle\boldsymbol{\xi},\mathbf{N}\rangle = \langle\mathbf{r}^*,\mathbf{N}\rangle = \frac{|\tilde{\nabla}\rho|}{\rho^2\sqrt{\rho^2 + |\tilde{\nabla}\rho|^2}} + \frac{\rho^2 - |\tilde{\nabla}\rho|^2}{2\rho^2\sqrt{\rho^2 + |\tilde{\nabla}\rho|^2}}$$

$$= \frac{\sqrt{\rho^2 + |\tilde{\nabla}\rho|^2}}{2\rho^2}.$$

Also, $\eta\langle\boldsymbol{\xi},\mathbf{N}\rangle\mathbf{N} = (\tilde{\nabla}\rho - \rho\mathbf{m})(2\rho)^{-1}$, and

$$\eta\mathbf{m} = \eta\boldsymbol{\xi} - (\tilde{\nabla}\rho - \rho\mathbf{m})\,\rho^{-2}, \tag{3.7}$$

Since $\mathbf{y} = \mathbf{m} - 2\,\langle\mathbf{m},\mathbf{n}\rangle\mathbf{n}$, we obtain from (1.1)

$$\eta\mathbf{y} = \eta\mathbf{m} - (\rho\mathbf{m} - \tilde{\nabla}\rho)\rho^{-2}, \tag{3.8}$$

Then from (3.7), (3.8) we get

$$\mathbf{y}(\mathbf{m}) = \boldsymbol{\xi}(\mathbf{m}) \text{ for all } \mathbf{m} \in \overline{\Omega}. \tag{3.9}$$

The proposition is proved.

3.3. It follows from (3.9) that $\gamma \circ \boldsymbol{\xi}^{-1} = \mathrm{Id}\colon \overline{\omega} \to \overline{\omega}$. Therefore,

$$J(\gamma) = J(\boldsymbol{\xi}).$$

This is also confirmed by a direct computation. Namely, it was shown

in [15] that

$$J(\xi) = M(p) = \frac{\det\left[\tilde{\nabla}_{ij}p + (p-\eta)e_{ij}\right]}{\eta^2 \det(e_{ij})}.$$

Then

$$M(1/\rho) = \frac{1}{\left(\rho^2 + |\tilde{\nabla}\rho|^2\right)^2} \frac{\det\left\{2\rho\tilde{\nabla}_{ij}\rho - \left(\rho^2 - |\tilde{\nabla}\rho|^2\right)e_{ij} - 4\rho_i\rho_j\right\}}{\det(e_{ij})}.$$

Comparing with (2.13), we see that

$$M(1/\rho) = G(\rho) \tag{3.10}$$

This formula will be useful in several instances.

The surface F^* is called the reciprocal reflector. With the use of complex structure on S, it has been described in [2] and {W], chapter 2, section 6.

4. Differential equations of the problem

4.1. First Order System. In this approach, followed by Galindo-Israel et al. [4-6] the analytic formulation of the problem is based on the following considerations.

The equation (1.8) written in component form is a system of three partial differential equations of the first order. However, since **m** and **y** are unit vectors, we may reduce (1.8) to a system of two equations as follows.

Observe that by (1.8)

$$\langle \mathbf{y},\mathbf{m}\rangle = 1 - 2\,\frac{\rho^2}{|\tilde{\nabla}\rho|^2 + \rho^2}. \tag{4.1}$$

Since $\rho > 0$, we may put $v := \ln\rho$. Then $v_k := \partial v/\partial u^k = \rho_k/\rho$ and we get

$$\langle \mathbf{y},\mathbf{m}\rangle = 1 - \frac{2}{1 + |\tilde{\nabla}v|^2} = \frac{|\tilde{\nabla}v|^2 - 1}{1 + |\tilde{\nabla}v|^2}. \tag{4.2}$$

Again from (1.8) we get

$$\langle \mathbf{y},\mathbf{m}_k\rangle = 2\frac{\rho\rho_k}{|\tilde{\nabla}\rho|^2 + \rho^2} = \frac{2v_k}{1 + |\tilde{\nabla}v|^2},\ k = 1,2. \tag{4.3}$$

We use (4.2) to solve for $|\tilde{\nabla}v|^2 + 1$. The result is

$$|\tilde{\nabla}v|^2 + 1 = \frac{2}{1 - \langle \mathbf{y},\mathbf{m}\rangle}.$$

Then we obtain from (4.3)

$$v_k = \frac{\langle \mathbf{y},\mathbf{m}_k\rangle}{1 - \langle \mathbf{y},\mathbf{m}\rangle},\ k = 1,2. \tag{4.4}$$

Thus, if both domains Ω and ω are given and we are given the vector field $\mathbf{y}(\mathbf{m})$, then by solving (4.4) for v we can ~~then~~ recover ρ and, therefore, the reflector surface F. However, for the system (4.4) to be solvable, an integrability condition must be satisfied [3], Ch. I. In this case, the condition is

$$v_{12} = v_{21} \tag{4.5}$$

where $v_{ik} = \partial^2 v/\partial u^i \partial u^k$, $i,k = 1,2$.

Therefore, for a reflector surface to exist, the following condition, derived from (4.4), must be satisfied.

$$(\langle \mathbf{y}_2,\mathbf{m}_2\rangle - \langle \mathbf{y}_2,\mathbf{m}_1\rangle)(1 - \langle \mathbf{y},\mathbf{m}\rangle) = \langle \mathbf{y},\mathbf{m}_2\rangle\langle \mathbf{y}_1,\mathbf{m}\rangle - \langle \mathbf{y},\mathbf{m}_1\rangle\langle \mathbf{y}_2,\mathbf{m}\rangle. \tag{4.6}$$

In a different form, this condition is given in [7] and in [4-6].

Following [5-6], one may formulate the problem of synthesizing a reflector as a question of solvability of the system (4.4) supplemented with equation (2.17) with prescribed $\overline{\Omega}$, $\overline{\omega}$, and functions I: $\overline{\Omega} \to (0,\infty)$ and V: $\overline{\omega} \to (0,\infty)$. Since in terms of the function v we have from (1.8)

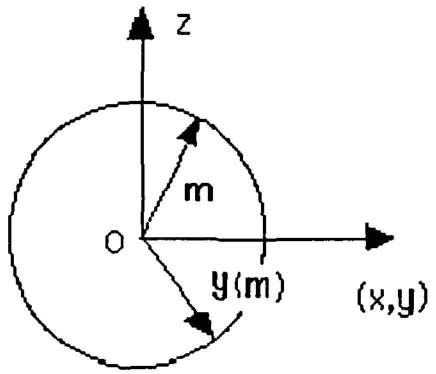
$$\mathbf{y}(\mathbf{m}) = \mathbf{m} - 2\,\frac{\mathbf{m} - \tilde{\nabla} v}{1 + |\tilde{\nabla} v|^2}, \tag{4.7}$$

the system (4.4) and equation (2.16) appear as a first order system of PDE's. However, as the formula (2.13) shows, the second derivatives are involved in the expression for the Jacobian.

In a series of papers, Galindo-Israel and his coauthors show that under some special circumstances, one can use heuristic arguments for constructing approximate solutions of this first order system . As mentioned earlier, their results rely on numerical calculations.

4.2. In some special circumstances, the integrability conditions (4.6) can be easily verified. For example, consider the case when Ω is a circular domain with center at the North pole of S and ω is also a circular domain with the center at the South pole. Let the vertical axis Ox pass through the North pole and (α,β) be spherical coordinates on S such that $0 \le \alpha \le \pi$, $0 \le \beta \le 2\pi$, with $\beta = 0$ corresponding to the positive direction of the z axis.

Suppose that the map γ is such that $\mathbf{y}(\mathbf{m})$ describes ω as a surface of revolution about the z axis and $\mathbf{y}(\mathbf{m})$ is obtained from $\mathbf{m}$ by rotating $\mathbf{m}$ in the plane passing through Oz and $\mathbf{m}$; see the fig. below.



We want to check that (4.6) is satisfied. Since $\mathbf{m}_\beta := \partial \mathbf{m}/\partial\beta$ is perpendicular to the plane spanned by **m** and the Oz axis, we have $<\mathbf{y}(\mathbf{m}),\mathbf{m}_\beta> = 0$. Because Ω and ω are rotationally symmetric about the Oz axis, $<\mathbf{y}(\mathbf{m}),\mathbf{m}> = f(\beta)$ for some function f. Then $<\mathbf{y}_\beta(\mathbf{m}),\mathbf{m}> = -<\mathbf{y}(\mathbf{m}),\mathbf{m}_\beta> = 0$ and substituting into the right hand side of (4.6) we obtain ($i = 1 \longleftrightarrow \alpha$, $k=2 \longleftrightarrow \beta$)

$$<\mathbf{y},\mathbf{m}_\beta><\mathbf{y}_\alpha,\mathbf{m}> - <\mathbf{y},\mathbf{m}_\alpha><\mathbf{y}_\beta,\mathbf{m}> = 0$$

Differentiating $<\mathbf{y},\mathbf{m}_\beta>$ in α, we obtain

$$0 = <\mathbf{y}_\alpha,\mathbf{m}_\beta> + <\mathbf{y},\mathbf{m}_{\alpha\beta}>.$$

But it is easy to check that $\mathbf{m}_{\alpha\beta}$ is perpendicular to **m** and to the Oz axis. Hence, it is perpendicular to $\mathbf{y}(\mathbf{m})$. Then $<\mathbf{y}_\alpha,\mathbf{m}_\beta> = 0$. Similarly, one checks that $<\mathbf{y}_\beta,\mathbf{m}_\alpha> = 0$. Thus (4.6) is satisfied.

In general, one does not know *a priori* the vector function $\mathbf{y}(\mathbf{m})$, but if such a function exists, it must satisfy (4.6).

4.3. The Monge-Ampère Equation. In this approach, one attempts to solve the equation (2.16). Let us formulate the problem precisely. Suppose we are given two domains $\overline{\Omega}$ and $\overline{\omega}$ on S and two positive functions I: $\overline{\Omega} \dashrightarrow (0,\infty)$ and V: $\overline{\omega} \to (0,\infty)$. The problem consists in finding a solution $\rho > 0$ of the equation

$$V(\mathbf{y}(\mathbf{m}))(G(\rho))(\mathbf{m}) = I(\mathbf{m}),\ \mathbf{m} \in \overline{\Omega}, \tag{4.8}$$

or

$$V(\mathbf{y}(\mathbf{m}))(G(\rho))(\mathbf{m}) = -I(\mathbf{m}),\ \mathbf{m} \in \overline{\Omega}, \tag{4.8'}$$

subject to the boundary condition

$$\gamma: \partial\Omega \to \partial\omega. \tag{4.9}$$

In (4.8) $\mathbf{y}(\mathbf{m})$ is expressed in terms of ρ as in (1.8). The boundary condition (4.9) is understood as the requirement that the boundary $\partial\Omega$ is mapped homeomorphically onto $\partial\omega$, but the map is not specified point-wise. One can show that if γ is specified point-wise on $\partial\Omega$, then the problem is overdetermined.

The data I, V, $\overline{\Omega}$, and $\overline{\omega}$ cannot be arbitrary since the energy conservation requirement (2.18) must be fulfilled. The latter can be rewritten as

$$\int_\omega V(\mathbf{y})d\sigma(\mathbf{y}) = \int_\Omega I(\mathbf{m})d\sigma(\mathbf{m}) \tag{4.10}$$

The operator $G(\rho)$ in (4.8) and (4.8)' is of, so called, Monge-Ampere type and whether it is elliptic or hyperbolic depends on the class of functions on which it is considered. More precisely, G will be positively (negatively) elliptic on any ρ for which the matrix

$$[2\rho\tilde{\nabla}_{ij}\rho - (\rho^2 - |\tilde{\nabla}\rho|^2)e_{ij} - 4\rho_i\rho_j] \equiv [a_{ij}(\rho)]$$

is positive (negative) definite. G will be hyperbolic on such ρ for which $[a_{ij}(\rho)]$ is indefinite but nondegenerate. Since $[a_{ij}(\rho)]$ is a 2×2 matrix, G is elliptic if and only if $\det[a_{ij}(\rho)] > 0$ in Ω and hyperbolic if $\det[a_{ij}(\rho)] < 0$ in Ω. Respectively, equation (4.8) corresponds to elliptic solutions, both positive elliptic and negative elliptic, and additional restrictions on ρ needs to be imposed in order to specify one of them.

The following simple example illustrates this situation. Take a

surface F given by the function $\rho = c$ =const. > 0. Then F is a sphere of radius c, $a_{ij}(c) = -c^2 e_{ij}$, $G(\rho) = 1$ (see (2.13)) and the matrix $[a_{ij}(c)]$ is negative definite.

Consider now a surface F which is plane tangent to S at the North Pole. One can calculate ρ and similarly compute the corresponding $[a_{ij}(\rho)]$. However, a simpler way is to observe from (2.12) and (2.13) that $[a_{ij}(\rho)](\rho^2 + |\tilde{\nabla}\rho|^2)^{-1} = [2b_{ij}\langle \mathbf{m},\mathbf{n}\rangle + \rho e_{ij}]\rho^{-1}$ and then recall that for a plane $b_{ij} \equiv 0$ [9]. Therefore, in this case $[a_{ij}(\rho)]$ is positive definite, since $[e_{ij}]$ is positive definite.

Finally, note that the equation (4.8)' corresponds to those ρ on which G is hyperbolic.

Thus, *a priori*, without considering G on a particular ρ, one cannot describe the type of G even though I/V is positive. We emphasize this standard point only because in [16] it was claimed that the corresponding equation is elliptic.

Let us now show that on elliptic "solutions" the Jacobian $J(\mathbf{m})$ is always positive. Indeed, if the map $\gamma: \Omega \to \omega$ preserves the orientation, the products $\langle \mathbf{m},\mathbf{m}_1 \times \mathbf{m}_2\rangle$ and $\langle \mathbf{y},\mathbf{y}_1 \times \mathbf{y}_2\rangle$ have the same sign. Since by our assumption $\langle \mathbf{m},\mathbf{m}_1 \times \mathbf{m}_2\rangle > 0$ in Ω, the product $\langle \mathbf{y},\mathbf{y}_1 \times \mathbf{y}_2\rangle > 0$ in Ω and consequently, $J(\mathbf{m}) > 0$ in Ω. In view of (1.5) and (2.13), $J(\mathbf{m})$ and $(G(\rho))(\mathbf{m})$ must have the same sign. Therefore, $(G(\rho))(\mathbf{m}) > 0$, which means that $J(\mathbf{m})$ is positive only on such ρ for which G is elliptic.

Similarly, one shows that $J(\mathbf{m}) < 0$ on those ρ on which G is hyperbolic.

5. Radially symmetric case

5.1. Now we consider the special case in which the reflector has axial symmetry. In this case, the PDE (4.8) reduces to an ordinary differential equation for which we can find explicit solvability conditions. It will be shown below that the situation here is very similar to the case considered in [15], and we follow this work closely.

It will be convenient in this case to use spherical coordinates α, β, where $-\pi/2 \le \alpha \le \pi/2$, $0 \le \beta \le 2\pi$. Assume that both domains Ω and ω

are circular with centers being respectively the North and South Poles.

$$\overline{\Omega} = \{ (\alpha,\beta) \mid \overline{\alpha} \leq \alpha \leq \pi/2\},\ \overline{\alpha} \in (0,\pi/2)\ ,$$

$$\overline{\omega} = \{ (\alpha,\beta) \mid -\pi/2 \leq \alpha \leq \tilde{\alpha}\},\ \tilde{\alpha} \in (-\pi/2,0).$$

In these coordinates,

$$\left[e_{ij}\right] = \begin{bmatrix} 1 & 0 \\ 0 & \cos^2\alpha \end{bmatrix}$$

We are looking for elliptic solutions of (4.8) in the form $\rho \equiv \rho(\alpha)$.

We introduce a new unknown function $p = 1/\rho$, assuming, as always, that $\rho > 0$. As it was shown in section 3, the operator $G(\rho) = G(1/p) = M(p)$. Using the expression for $M(p)$ in the axially symmetric case as in [15], section 3, we obtain from (4.8)

$$V(\mathbf{y}(\mathbf{m}))M(p) \equiv V(\mathbf{y}(\mathbf{m}))\,\frac{(2pp + p^2 - p^2)(-2pp\tan\alpha + p^2 - p)}{(\dot{p}^2 + p^2)^2} = I(\mathbf{m}), \qquad (5.1)$$

$$\overline{\alpha} < \alpha < \pi/2,$$

where $\dot{p} = dp/d\alpha$ $\dot{p} = d^2p/d\alpha^2$.

In order for $\rho(\alpha)$ to be smooth in Ω, we need to impose the condition

$$\dot{\rho}(\pi/2) = 0 \qquad (5.2)$$

which is equivalent to

$$\dot{p}(\pi/2) = 0. \qquad (5.3)$$

We now set up the boundary conditions. This amounts to describing (4.9) in analytic form. If one prescribes γ: $\partial\Omega \to \partial\omega$ point-wise, then the problem is overdetermined (cf. [5], [6], [7], [8]). A way to relax this restriction is to require that γ maps $\partial\Omega$ onto $\partial\omega$ homeomorphically, but without specifying the map point-wise.

Thus, if $\mathbf{y} = \gamma(\mathbf{m})$, $\mathbf{m} \in \partial\Omega$, then

$$\langle \mathbf{m},(0,0,1)\rangle = \cos(\pi/2 - \overline{\alpha}) = \sin\overline{\alpha} \quad \text{on } \partial\Omega,$$

$$\langle \mathbf{y},(00,-1)\rangle = \cos(\pi/2 - \tilde{\alpha}) = -\sin\tilde{\alpha} \quad \text{on } \partial\omega.$$

On the other hand, by (1.8) we have, taking into account that $|\tilde{\nabla}p|^2 = \dot{p}^2$,

$$\mathbf{y} = \mathbf{m} - \frac{2p\left(p\mathbf{m} + \tilde{\nabla}p\right)}{\dot{p}^2 + p^2}.$$

Then

$$-\sin\tilde{\alpha} = -\sin\overline{\alpha} + \frac{2p^2}{\dot{p}^2 + p^2}\sin\overline{\alpha} - \frac{2p}{\dot{p}^2 + p^2}\langle\tilde{\nabla}p,(0,0,-1)\rangle .$$

Since $\tilde{\nabla}p = \dot{p}\,\partial\mathbf{m}/\partial\alpha$, we have $\langle\tilde{\nabla}p,(0,0,-1)\rangle = -\dot{p}\cos\overline{\alpha}$ and

$$-\sin\tilde{\alpha} = \frac{p^2 - \dot{p}^2}{p^2 + \dot{p}^2}\sin\overline{\alpha} + \frac{2p\dot{p}}{p^2 + \dot{p}^2}\cos\overline{\alpha}$$

or,

$$\dot{p}^2(\sin\tilde{\alpha} - \sin\overline{\alpha}) = p^2(\sin\tilde{\alpha} + \sin\overline{\alpha}) + 2p\dot{p}\cos\overline{\alpha} = 0 \qquad (5.4)$$

where p and $\dot{p}$ are evaluated at $\alpha = \overline{\alpha}$. This boundary condition is identical with (2.3) in [15]. In the special case when the illumination pattern is required to be uniform, that is, $V(\mathbf{y}(\mathbf{m})) = \text{const} \equiv V_0$, then the problem (5.1), (5.3), and (5.4) is analytically the same as the one in [15] and we can formulate the following result (cf. Theorem 3.5 in [15]).

Theorem 5.2. Let $I(\alpha)$ be positive and continuous on $[0,\pi/2]$. Let $\tilde{\alpha}$ be any number in the interval $(-\pi/2,0)$ and $\overline{\alpha}$ the solution of the equation

$$1 + \sin\tilde{\alpha} = \int_{\overline{\alpha}}^{\frac{\pi}{2}} I(\tau)\cos\tau\, d\tau, \quad \overline{\alpha} \in (0,\pi/2) .$$

Then for each choice of $p(\overline{\alpha})$, $\dot{p}(\overline{\alpha})$ such that

$$\frac{\dot{p}(\overline{\alpha})}{p(\overline{\alpha})} = \frac{\overline{\alpha}}{2} - \frac{\tilde{\alpha}}{2}$$

or

$$\frac{\dot{p}(\overline{\alpha})}{p(\overline{\alpha})} = \frac{\pi}{2} - \frac{\overline{\alpha}}{2} + \frac{\tilde{\alpha}}{2}$$

there exists a unique solution $p > 0$ of (5.1), (5.3), (5.4) of class $C^1[\overline{\alpha},\pi/2] \cap C^2(\overline{\alpha},\pi/2]$.

5.3. When the function p is constructed, then we return to the function $\rho = 1/p$ and consider $\mathbf{r}(\mathbf{m}) = \rho(\alpha,\beta)\mathbf{m}(\alpha,\beta)$ where we set $\rho(\alpha,\beta) \equiv \rho(\alpha)$. The vector function $\mathbf{r}$ defines the reflector surface. Clearly, the map γ is defined and $J(\mathbf{m}) > 0$. Thus γ is a local diffeomorphism.

6. An algorithm for computing the output power density.

It follows from formulas (2.16) and (2.12) that the computation of the output power intensity reduces to computation of the Jacobian of the map γ. In this section we provide a discretization of (2.12) and several computational examples. We preserve here the notation from previous sections.

ALGORITHM FOR THE COMPUTATION OF $G(\rho)$

Let $U = \{(\alpha,\beta): \bar{\alpha} \le \alpha \le \frac{\pi}{2}, 0 \le \beta \le 2\pi\}$, where $-\frac{\pi}{2} < \bar{\alpha} < \frac{\pi}{2}$, and let $\bar{\Omega} = \{\mathbf{m}(\alpha,\beta) = \cos(\alpha)\cos(\beta), \cos(\alpha)\sin(\beta), \sin(\alpha): (\alpha,\beta) \in U\}$. If ρ is a nonnegative function on U, let $F = \{\mathbf{r}(\alpha,\beta) = \rho(\alpha,\beta)\mathbf{m}(\alpha,\beta): (\alpha,\beta) \in U\}$ be a reflector and let $G(\rho)$ be the Jacobian of the reflector map γ as in (2.12). We give an algorithm for computing $G(\rho)$ at the points $(\bar{\alpha}+sh, lt)$ in U, where

$$h = \frac{\frac{\pi}{2} - \bar{\alpha}}{\bar{s}-1},\ k = \frac{2\pi}{\bar{l}},\ s = 1,\dots,\bar{s}-1 \text{ and } l = 0,\dots,\bar{l}-1.$$

Note: we do not compute $G(\rho)$ on the boundary of F.

In the following we write v_1 for $\frac{\partial v}{\partial \alpha}$ and v_2 for $\frac{\partial v}{\partial \beta}$.

Algorithm:

1. Approximate the tangent vectors to the sphere, $S_1(\bar{\alpha}+sh, lk)$ and $S_2(\bar{\alpha}+sh, lk)$, at $\mathbf{m}(\bar{\alpha}+sh, lk)$ and the tangent vectors to F, $T_1(\bar{\alpha}+sh, lk)$ and $T_2(\bar{\alpha}+sh, lk)$, at $\mathbf{r}(\bar{\alpha}+sh, lk)$ in one of 3 ways:

   a. if $s = \bar{s}-1$ (i.e. if $\mathbf{m}(\bar{\alpha}+sh, lk) = (0,0,1)$) let

$$S_1(\bar{\alpha}+sh, lk) = \frac{\mathbf{m}(\bar{\alpha}+(s-1)h, (\frac{\bar{l}}{2}+l \bmod(\bar{l}))k) - \mathbf{m}(\bar{\alpha}+(s-1)h, lk)}{2h},$$

$$S_2(\bar{\alpha}+sh, lk) = \frac{\mathbf{m}(\bar{\alpha}+(s-1)h, (\frac{\bar{l}}{4}+l \bmod(\bar{l}))k) - \mathbf{m}(\bar{\alpha}+(s-1)h, (\frac{3\bar{l}}{4}+l \bmod(\bar{l}))k)}{2k},$$

$$T_1(\bar{\alpha}+sh, lk) = \frac{\mathbf{r}(\bar{\alpha}+(s-1)h, (\frac{\bar{l}}{2}+l \bmod(\bar{l}))k) - \mathbf{r}(\bar{\alpha}+(s-1)h, lk)}{2h},$$

$$T_2(\bar{\alpha}+sh, lk) = \frac{\mathbf{r}(\bar{\alpha}+(s-1)h, (\frac{\bar{l}}{4}+l \bmod(\bar{l}))k) - \mathbf{r}(\bar{\alpha}+(s-1)h, (\frac{3\bar{l}}{4}+l \bmod(\bar{l}))k)}{2k}.$$

   b. if $0 < s < \bar{s}-1$

$$S_1(\bar{\alpha}+sh, lk) = \frac{\mathbf{m}(\bar{\alpha}+(s-1)h, lk) - \mathbf{m}(\bar{\alpha}+(s+1)h, lk)}{2h},$$

$$S_2(\bar{\alpha}+sh, lk) = \frac{\mathbf{m}(\bar{\alpha}+sh, l+1) - \mathbf{m}(\bar{\alpha}+sh, l-1)}{2k},$$

$$T_1(\bar{\alpha}+sh,lk) = \frac{\mathbf{r}(\bar{\alpha}+(s-1)h,lk) - \mathbf{r}(\bar{\alpha}+(s+1)h,lk)}{2h},$$

$$T_2(\bar{\alpha}+sh,lk) = \frac{\mathbf{r}(\bar{\alpha}+sh,l+1) - \mathbf{r}(\bar{\alpha}+sh,l-1)}{2k}.$$

c. if $s = 0$ (i.e. $\mathbf{m}(\bar{\alpha}+sh,lk)$ lies on the boundary of $\bar{\Omega}$) we make a one-sided approximation of $S_1(\bar{\alpha}+sh,lk)$ and $T_1(\bar{\alpha}+sh,lk)$:

$$S_1(\bar{\alpha}+sh,lk) = \frac{\mathbf{m}(\bar{\alpha}+(s+1)h,lk) - \mathbf{m}(\bar{\alpha}+sh,lk)}{h},$$

$$S_2(\bar{\alpha}+sh,lk) = \frac{\mathbf{m}(\bar{\alpha}+sh,l+1) - \mathbf{m}(\bar{\alpha}+sh,l-1)}{2k},$$

$$T_1(\bar{\alpha}+sh,lk) = \frac{\mathbf{r}(\bar{\alpha}+(s+1)h,lk) - \mathbf{r}(\bar{\alpha}+sh,lk)}{h},$$

$$T_2(\bar{\alpha}+sh,lk) = \frac{\mathbf{r}(\bar{\alpha}+sh,l+1) - \mathbf{r}(\bar{\alpha}+sh,l-1)}{2k}.$$

Remark: $T_1(\bar{\alpha},lk)$ best approximates a tangent vector at the $\mathbf{r}(\bar{\alpha}+\frac{h}{2},lk)$.

2. For $0 \le s \le \bar{s}-1$ and $1 \le l \le \bar{l}-1$ estimate $\mathbf{n}(\bar{\alpha}+sh,lk)$ as $\dfrac{T_1(\bar{\alpha}+sh,lk)\times T_2(\bar{\alpha}+sh,lk)}{|T_1(\bar{\alpha}+sh,lk)\times T_2(\bar{\alpha}+sh,lk)|}$.

3. We approximate the directional derivatives of $\mathbf{n}(\bar{\alpha}+sh,lk)$ with respect to $T_1(\bar{\alpha}+sh,lk)$ and $T_2(\bar{\alpha}+sh,lk)$ once again in 3 ways:

a. if $s = \bar{s}-1$

$$\mathbf{n}_1(\bar{\alpha}+sh,lk) = \frac{\mathbf{n}(\bar{\alpha}+(s-1)h,(\frac{\bar{l}}{2}+l \bmod(\bar{l}))k) - \mathbf{n}(\bar{\alpha}+(s-1)h,lk)}{2h},$$

$$\mathbf{n}_2(\bar{\alpha}+sh,lk) = \frac{\mathbf{n}(\bar{\alpha}+(s-1)h,(\frac{\bar{l}}{4}+l \bmod(\bar{l}))k) - \mathbf{n}(\bar{\alpha}+(s-1)h,(\frac{3\bar{l}}{4}+l \bmod(\bar{l}))k)}{2k}.$$

b. if $1 < s < \bar{s}-1$

$$\mathbf{n}_1(\bar{\alpha}+sh,lk) = \frac{\mathbf{n}(\bar{\alpha}+(s-1)h,lk) - \mathbf{n}(\bar{\alpha}+(s+1)h,lk)}{2h},$$

$$\mathbf{n}_2(\bar{\alpha}+sh,lk) = \frac{\mathbf{n}(\bar{\alpha}+sh,l+1) - \mathbf{n}(\bar{\alpha}+sh,l-1)}{2k}.$$

c. if $s = 1$, we use the remark after step 1c to conclude that since $T_1(\bar{\alpha},lk)$ best approximates the tangent vector at $\mathbf{r}(\bar{\alpha}+\frac{h}{2},lk)$, $\mathbf{n}(\bar{\alpha},lk)$ best approximates the normal at

$\mathbf{r}(\bar{\alpha}+\frac{h}{2},lk)$:

$$\mathbf{n}_1(\bar{\alpha}+sh,lk) = \frac{\mathbf{n}(\bar{\alpha}+(s-1)h,lk) - \mathbf{n}(\bar{\alpha}+(s+1)h,lk)}{1.5h},$$

$$\mathbf{n}_2(\bar{\alpha}+sh,lk) = \frac{\mathbf{n}(\bar{\alpha}+sh,(l+1)k) - \mathbf{n}(\bar{\alpha}+sh,(l-1)k)}{2k}.$$

4. Approximate the metric at $\mathbf{m}(\bar{\alpha}+sh,lk)$ with respect to $S_1(\bar{\alpha}+sh,lk)$ and $S_2(\bar{\alpha}+sh,lk)$ as

   $e_{ij}(\bar{\alpha}+sh,lk) = \langle S_i(\bar{\alpha}+sh,lk), S_j(\bar{\alpha}+sh,lk)\rangle$ for $i = 1,2$ and $j = 1,2$.

5. Approximate the second fundamental form at $\mathbf{r}(\bar{\alpha}+sh,lk)$ as

   $b_{ij}(\bar{\alpha}+sh,lk) = -\langle \mathbf{n}_i(\bar{\alpha}+sh,lk), T_j(\bar{\alpha}+sh,lk)\rangle$ for $i = 1,2$ and $j = 1,2$.

6. Evaluate $G(\rho(\bar{\alpha}+sh,lk))$ for $0 \le s \le \bar{s}-2$ and $0 \le l \le \bar{l}-1$ as

$$\det(e_{ij}(\bar{\alpha}+sh,lk))^{-1} \det\left(2\frac{b_{ij}(\bar{\alpha}+sh,lk)\langle \mathbf{m}(\bar{\alpha}+sh,lk),\mathbf{n}(\bar{\alpha}+sh,lk)\rangle}{\rho(\bar{\alpha}+sh,lk)} + e_{ij}(\bar{\alpha}+sh,lk)\right).$$

We used this algorithm for examples where $G(\rho)$ is known. These were for $\rho(\alpha,\beta) = 1$ (the unit sphere with the center at the origin), for $\rho(\alpha,\beta) = \csc(\alpha)$ (the plane $z = 1$) and for $\rho(\alpha,\beta) = \frac{2}{1+\sin(\alpha)}$ (the paraboloid with equation $z = 1 - \frac{1}{4}(x^2 + y^2)$ with its focus at the origin).

For all three surfaces we took $\bar{\alpha} = \frac{\pi}{4}$, $\bar{s} = 25$ and $\bar{l} = 24$.

Example 1: For the unit sphere $G(\rho) \equiv 1$. Our algorithm had a maximum error of 8.9×10^{-5}, which occured for $s = 1$.

Example 2: For the plane $G(\rho) \equiv 1$. Our algorithm computed this with no error since it computed the second fundamental form as precisely 0.

Example 3: For the paraboloid $G(\rho) \equiv 0$. Our algorithm had a maximum error of 7.6×10^{-7}, which occured for $s = 1$.

We have also used this algorithm on offset examples.

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On one direct problem in the reflector antenna theory

by

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Abstract

This paper is the second in a series of papers in which we use differential geometric methods to investigate systematically the problem of synthesis of single and dual reflector antennas. In our first paper we considered the direct problem for a single reflector and, in particular, established rigorously existence of radially symmetric solutions in the case when the data is radially symmetric. In present article we prove existence of reflectors solving the direct problem in the case when the data is not radially symmetric but close to such in some Holder norm.

1. Introduction

This paper is the second in a series of papers in which we study the theory and numerical methods in synthesis of reflector antennas. We use the geometric optics (GO) approximation to describe and study the problem. A brief history of the work of other researchers' in constructing reflector antennas using GO can be found in our paper [ONP].

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Building upon our previous work [ONP] and [O], we show now existence of nonradially symmetric solutions do direct problem in the case where the density of distribution of reflected rays is uniform and the density of the distribution of the incidence rays is not radially symmetric but close in a certain norm to a radially symmetric function.

2. Preliminaries and the main result.

2.1 We begin by recalling the notation used in [ONP]. In three dimensional space $\mathbf{R}^3$ let S be a unit sphere centered at the origin O of a Cartesian coordinate system. Let Ω be a domain on S, $\overline{\Omega}$ its closure, and $\mathbf{m}$ a unit vector with endpoint in Ω. Let ρ be a positive function of class $C^2(\Omega)\cap C^1(\overline{\Omega})$ and set

$$\mathbf{r}(\mathbf{m}) = \rho(\mathbf{m})\mathbf{m}. \tag{1}$$

The map $\mathbf{r}$ defines *a surface F projecting radially from O univalently* onto $\overline{\Omega}$. Denote by $\mathbf{n}$ the unit normal vector field on F and assume that F is oriented so that $\langle \mathbf{m},\mathbf{n}\rangle > 0$ everywhere on F. Here, $\langle\ ,\ \rangle$ denotes the usual scalar product in $\mathbf{R}^3$.

If a light ray originates at O in the direction $\mathbf{m}$, reaches F, (the reflector) and is reflected, then we can put in correspondence with $\mathbf{m}$ a unit vector $\mathbf{y} \in S$ parallel to the reflected ray. Thus, we have the "reflector" map γ: $\overline{\Omega}$-->S, $\mathbf{y} = \gamma(\mathbf{m})$. The laws of geometric optics tell us that the vectors $\mathbf{m}$, $\mathbf{n}(\mathbf{m})$, and $\gamma(\mathbf{m})$ lie in one plane and $\gamma(\mathbf{m}) = \mathbf{m} - 2\langle\mathbf{m},\mathbf{n}\rangle\mathbf{n}$. The image $\overline{\omega} = \gamma(\overline{\Omega})$ is called the "far field".

Let $u = (u^1,u^2)$ be some smooth local coordinates on S such that $\overline{\Omega}$ lies inside one coordinate patch (it is always assumed that $\overline{\Omega} \neq S$). Then $\mathbf{m}(u) \equiv \mathbf{m}(u^1,u^2)$ *is a smooth vector valued function giving the position* vector of any point in $\overline{\Omega}$. For that reason $\mathbf{m}(u)$ is viewed as a unit vector

in $\mathbf{R}^3$ and also as a point in $\overline{\Omega}$. As usual, we put $f(\mathbf{m}(u)) \equiv f(u)$ for any function $f: S \to \mathbf{R}$. Everywhere in the paper we use the range of indices $1 \le i,j,k,.. \le 2$.

The first fundamental form $e = e_{ij}du^i du^j$ of S has coefficients $e_{ij} = \langle \mathbf{m}_i, \mathbf{m}_j \rangle$ where $\mathbf{m}_i = \partial \mathbf{m}/\partial u^i$. Here and everywhere below the convention about summation over repeated lower and upper indices is in effect. The matrix $[e_{ij}]$ is symmetric and invertible; its inverse is denoted by $[e^{ij}]$.

Set

$$\tilde{\nabla}\rho = \rho_i e^{ij} \mathbf{m}_j.$$

Then

$$|\tilde{\nabla}\rho|^2 = \langle \tilde{\nabla}\rho, \tilde{\nabla}\rho \rangle = \rho_i \rho_j e^{ij}.$$

It is shown in [ONP] that for any $\rho \in C^1(\overline{\Omega})$, $\rho > 0$ in $\overline{\Omega}$, the vector field

$$\mathbf{n} = \frac{\rho \mathbf{m} - \tilde{\nabla}\rho}{\sqrt{\rho^2 + |\tilde{\nabla}\rho|^2}} \tag{2}$$

is the unit normal vector field on the surface F defined by (1). In addition, $\langle \mathbf{r}, \mathbf{n} \rangle > 0$ on F. Further, if $\mathbf{y} = \gamma(\mathbf{m})$ is the unit vector parallel to the reflected ray corresponding to $\mathbf{m}$ then

$$\mathbf{y} = \mathbf{m} - 2\frac{\rho(\rho \mathbf{m} - \tilde{\nabla}\rho)}{\rho^2 + |\tilde{\nabla}\rho|^2} \tag{3}$$

2.2 In the *direct reflector problem* the following data is given: the domains $\overline{\Omega}$, $\overline{\omega}$ on S, a positive function $I: \Omega \to (0,\infty)$, and a positive constant V_0. We have to find a reflector F subject to the requirements:

(i) the rays originating at O and going through points of $\overline{\Omega}$ project F univalently onto $\overline{\Omega}$;

(ii) $\overline{\omega}$ is the far field;

(iii) for the given density $I(\mathbf{m})$ of the incidence rays the density of the distribution of rays reflected of F in the direction $\gamma(\mathbf{m})$ is uniform and equal to V_0;

(iv) finally, it is also natural to require, in this setting, that F be such that the reflector map γ is a diffeomorphism of $\overline{\Omega}$ onto $\overline{\omega}$.

It is shown in [ONP] that in terms of function ρ the Jacobian of the map γ is given by

$$G(\rho) := \frac{1}{(\rho^2 + |\tilde{\nabla}\rho|^2)} \frac{\det [2\rho\tilde{\nabla}_{ij}\rho - (\rho^2 - |\tilde{\nabla}\rho|^2)e_{ij} - 4\rho_i\rho_j]}{\det [e_{ij}]}. \tag{4}$$

Thus, in order to find a reflector F satisfying (iii) we need to solve the PDE (cf. [ONP], subsection 2.5)

$$G(\rho(\mathbf{m})) = I(\mathbf{m})/V_0, \quad \text{in } \Omega. \tag{5}$$

If the reflector map γ, determined by F, satisfies (i) and (iv) then it is necessary that

$$\gamma: \partial\Omega \dashrightarrow \partial\omega. \tag{6}$$

By (2), γ is expressed in terms of ρ, and, therefore, (6) is a condition on ρ on $\partial\Omega$. Hence, we may view it as a boundary condition to be satisfied by solutions of (5). However, if (6) is understood as a pointwise condition, then the problem (5), (6) is overdetermined and, in general, will not have a solution. For this reason (6) is treated as a requirement that the boundary of Ω be mapped onto the boundary of ω.

2.3 In order to formulate our main result we need some more notation.

Let (α,β), $-\pi/2 \le \alpha \le \pi/2$, $0 \le \beta \le 2\pi$, be the spherical coordinates on S. Let

$$\overline{\Omega} = \{(\alpha,\beta) \mid \overline{\alpha} \le \alpha \le \pi/2\},\ \overline{\alpha} \in (0,\pi/2),$$

$$\overline{\omega} = \{(\alpha,\beta) \mid -\pi/2 \le \alpha \le \tilde{\alpha}\},\ \tilde{\alpha} \in (-\pi/2, 0).$$

Let $\overline{I}$ be a positive function of class $C^1[0,\pi/2]$. Put

$$E(\alpha_1,\alpha_2) = \int_{\alpha_1}^{\alpha_2} \overline{I}(\tau)\cos\tau\, d\tau.$$

For $\delta \in (0,1)$ put

$$H = \{ I \in C^{0,\delta}(\overline{\Omega}) \mid \int_{\Omega} I\, d\sigma = 2\pi V_0(1 + \sin\tilde{\alpha})\}, \qquad (7)$$

where $d\sigma$ is the area element of S.

Theorem A. Suppose that Ω, ω and $\overline{I}$ are such that

$$E(\overline{\alpha}, \pi/2) = V_0(1 + \sin\tilde{\alpha}). \qquad (8)$$

Then there exists an $\epsilon > 0$ such that for any $I \in H$ satisfying $\| I - \overline{I} \| < \epsilon$, where $\| \ \|$ denotes the Holder norm in $C^{0,\delta}(\overline{\Omega})$, the equation

$$G(\rho) = I/V_0 \quad \text{in } \Omega \qquad (9)$$

with the boundary condition

$$\langle \gamma(\mathbf{m}),(0,0,-1)\rangle \big|_{\partial\Omega} = \text{geodesic radius of } \overline{\omega} \qquad (10)$$

admits two classes of solutions $\rho \in C^{2,\delta}(\overline{\Omega})$. Within each class the solution is determined uniquely up to a positive multiplicative constant. Furthermore, the corresponding reflector surfaces satisfy the conditions (i)-(iv) as described previously.

Remarks.

(a) The condition $I \in H$ together with (7) expresses the requirement that the data must satisfy the energy conservation law (cf. [ONP], subsection 2.3).

(b) If for $\overline{\alpha} \le \alpha \le \pi/2$, $0 \le \beta \le 2\pi$, we set $\overline{I}(\alpha,\beta) \equiv \overline{I}(\alpha)$ then (8) implies that $\overline{I} \in H$. It is shown in [ONP], Theorem 5.1, that for Ω, ω, $\overline{I}$, and V_0 satisfying (8) one can construct two classes of radially symmetric solutions to (9), (10) (with $\overline{I}$ on the right hand side of (9)) and in each class two solutions differ at most by a positive multiplicative constant. The role of the function $\overline{I}$ and any of the corresponding radially symmetric solutions is that these special solutions serve as approximations to nonradially symmetric solutions of (9), (10). In this article this connection remains in the background and is not explicitly shown, since the proof of Theorem A is based on a reduction of the direct problem to the inverse problem investigated in [O]. For the inverse problem this connection is explained in detail in [O].

(c) Let $I \in H \cap C^1(\overline{\Omega})$. Put

$$I_m(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} I(\alpha,\beta)d\beta.$$

It is easy to see that the function I_m satisfies all the conditions imposed on $\overline{I}$. Thus, the hypothesis regarding existence of the function $\overline{I}$ can be replaced by the requirement that $\|I - I_m\| < \epsilon$.

3. Proof of Theorem A

Step 1. First we show that there exist two distinct classes of

functions satisfying (9), (10). Following [ONP], section 3, we introduce a new unknown function $p = 1/\rho$. Then (9) becomes

$$M(p) := \frac{\det [\tilde{\nabla}_{ij} p + (p-\eta)e_{ij}]}{\eta^2 \det [e_{ij}]} = \frac{I}{V_0} \quad \text{in } \Omega, \tag{11}$$

where $\eta = (p^2 + |\tilde{\nabla} p|^2)/2p$. Since (10) is not written explicitly in terms of ρ, it will remain the same, but in order to emphasize that we are using the function p, we rewrite it as

$$\langle \gamma_p(\mathbf{m}),(0,0,-1)\rangle |_{\partial\Omega} = \text{geodesic radius of } \overline{\omega}. \tag{12}$$

With any $p > 0$, which is a solution of (11), (12), we associate the reciprocal reflector F^*, that is, the surface

$$-\mathbf{r}^* = \tilde{\nabla} p + (p-\eta)\mathbf{m}, \quad \mathbf{m} \in \overline{\Omega}.$$

We check by a direct computation that $|\mathbf{r}^*| = \eta$ and $\boldsymbol{\xi} := \mathbf{r}^*/\eta = \gamma_p$ in $\overline{\Omega}$. Thus, the problem (11), (12) is identical to the problem (3.1), (3.2) in [O] (with the roles of Ω and ω interchanged). On the other hand, if in Theorems 2.1 and 3.1 in [O] we set $\overline{f} = \overline{I}/V_0$ and interchange the roles of $\overline{\alpha}$ and $\tilde{\alpha}$ then the hypotheses of our Theorem A imply the hypotheses of Theorems 2.1 and 3.1 in [O]. Consequently, we conclude that the problem (11), (12) admits two classes of positive solutions in $C^2(\overline{\Omega})$ and within each class the solution is unique up to a positive multiplicative constant. Furthermore, in case $I \equiv \overline{I}$ the solutions of (11), (12) are radially symmetric.

We return now to the original unknown function $\rho = 1/p$ and that completes the step 1 of the proof.

Step 2. Let ρ be a solution of (9), (10) and F the surface defined by (1). On this step we check the conditions (i)-(iv) in subsection 2.2.

It follows from the discussion in sections 2.1 and 2.2 that F is a reflector surface with the normal field given by (2) and the far field described by vectors $y = \gamma(\mathbf{m})$ given by (3). The condition (i) is obviously satisfied. Condition (iii) is satisfied because (9) is satisfied. It remains to check (ii) and (iv).

Since the Jacobian $J(\gamma) = G = I/V_0 > 0$ in $\overline{\Omega}$, the map γ is a local diffeomorphism. Because of (10) γ maps $\overline{\Omega}$ into $\overline{\omega}$. By Corrolary 4.7 in [KN], since $\overline{\Omega}$ is compact, the map γ is a covering projection. On the other hand, if γ is not a global diffeomorphism of $\overline{\Omega}$ onto $\overline{\omega}$ then, since $J(\gamma) > 0$,

$$\text{area of } \omega = 2\pi(1 + \sin\tilde{\alpha}) < \int_\Omega J(\gamma)d\sigma = \int_\Omega G(\rho)d\sigma = \frac{1}{V_0}\int_\Omega I\, d\sigma.$$

The latter contradicts the hypothesis that $I \in H$. Now the conditions (ii) and (iv) are also verified and the theorem is proved.

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